Homework 3 Solutions

Math 131B-1

- (3.38) Let $S \subset T \subset M$. Recall that a set U is open in T if and only if it is equal to the intersection $V \cap T$ of T with some set V which is open in M. With that in mind, first suppose S is compact in M. Then if $\{U_\alpha\}_{\alpha \in A}$ is a covering of S consisting of sets in T, we may find an open covering $\{V_\alpha\}_{\alpha \in A}$ consisting of open sets $V_\alpha \subset M$ such that $V_\alpha \cap T = U_\alpha$. Since S is compact in M, some finite subcover $\{V_1, \cdots, V_n\}$ covers S. So $S \subset \bigcap_{i=1}^n V_i$, so since $S \subset T$, $S \subset \bigcap_{i=1}^n (V_i \cap T) = \bigcap_{i=1}^n U_i$. Therefore $\{U_1, \cdots, U_n\}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in A}$, and since $\{U_\alpha\}_{\alpha \in A}$ was an arbitrary open covering of S in T, S is compact in T. The other direction, proving that S compact in T implies S compact in M, is very similar.

  This shows that compactness is an absolute, not relative, property.

- (3.42) Let $(M, d) = (\mathbb{Q}, |\cdot|)$, that is, our ambient metric space is the rational numbers with the usual notion of distance. Let $S = (a, b) \cap \mathbb{Q}$. Then S is clearly bounded since $S \subset B(a; |b - a|)$. Moreover S is closed, since S is the intersection of the closed set $[a, b]$ in $(\mathbb{R}, |\cdot|)$ with M. However, we claim that S is not compact. We will prove this by giving an infinite subset of S which does not have a limit point. For every $n \in \mathbb{N}$, let c_n be a rational number in $(a, a + \frac{1}{n}) \cap (a, b)$. Then $C = \{c_n\} \subset S$ has only one limit point in $\mathbb{R}$, namely a, and no limit points in S. (To see this, observe that if x is a limit point of C, then any neighbourhood of x must contain infinitely many points of C. This is only true of a.) Therefore S is not compact.

- (3.17) Let $S \subset \mathbb{R}^n$. For each isolated point x of S, choose a ball $U_x = B(x; r)$ which contains no points of x other than x. Also, let $U' = \mathbb{R}^n - T$, where T is the set of all isolated points of S. Then $U' \bigcup \{U_x : x \in T\}$ is an open cover of S. Ergo a countable subcover also covers S. But since every set in our cover contains at most one point of T, this implies T is countable.

- (3.20) Let $S = \mathbb{Z} \subset \mathbb{R}$, and let $U_z = (z - \frac{1}{2}, z + \frac{1}{2})$ for all $z \in \mathbb{Z}$. Then $\{U_z : z \in \mathbb{Z}\}$ is a countable open covering of $\mathbb{Z}$ such that no finite subcover covers $\mathbb{Z}$ (since every open set in the cover contains exactly one integer).

- (3.22) A collection of disjoint open sets $\{U_\alpha\}$ in $\mathbb{R}^n$ is an open cover for itself. Since we know any open cover of a set in $\mathbb{R}^n$ can always be reduced to a countable subcover, and no nontrivial subcover of $\{U_\alpha\}$ covers $S = \bigcup U_\alpha$, we see that the cover $\{U_\alpha\}$ must have been countable to start with. For the closed case, consider $\{\{r\} : r \in \mathbb{R}\}$, which is an uncountable collection of closed disjoint sets.

- (4.7) Let $x_n \to x$ and $y_n \to y$ in (S, d). Let $\epsilon > 0$. Then there exists N_1 such that for $n > N_1$, $d(x_n, x) < \frac{\epsilon}{2}$ and N_2 such that for $n > N_2$, $d(y_n, y) < \frac{\epsilon}{2}$. By the triangle inequality, for any n we know that $d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n)$, implying that $d(x_n, y_n) - d(x, y) < d(x_n, x) + d(y, y_n)$, and similarly, $d(x, y) \le d(x, x_n) + d(x_n, y_n) + d(y_n, y)$, so $d(x, y) - d(x_n, y_n) < d(x_n, x) + d(y, y_n)$. Therefore $|d(x, y) - d(x_n, y_n)| < d(x_n, x) + d(y_n, y)$, and in particular, for any $n > \max\{N_1, N_2\}$, we have $|d(x, y) - d(x_n, y_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Ergo $d(x_n, y_n) \to d(x, y)$.

- (4.8) Let (S, d) be a compact metric space. Let $\{x_n\}$ be a sequence in S, and $T = \{x_n : n \in \mathbb{N}\}$ be its set of values. If T is finite, $\{x_n\}$ is eventually constant, hence converges. If T is infinite, T has a limit point p in S. Choose some $\{x_{k_1}\}$ such that $x_{k_1} \in B(p; 1)$; this will be the first element of a subsequence converging to p. Assume we have constructed the $(n-1)$st element $x_{k_{n-1}}$ of our subsequence. Then to construct the nth element, choose some $x_{k_n} \in B(p; \frac{1}{n})$ such that $k_n > k_{n-1}$. This is always possible since there are infinitely many points of T in $B(p; \frac{1}{n})$ but only finitely many points x_n of T such that $n < k_{n-1}$. The subsequence $\{x_{k_n}\}$ we have chosen converges to p by construction.

- (4.9)Let A be a complete subspace of a metric space S. Suppose that p is a limit point of S. Then by Theorem 4.4, there is a sequence $\{x_n\}$ in A converging to p; since $\{x_n\}$ is convergent, it must be Cauchy, hence must converge in A. This implies that $p \in A$. Ergo the set A contains all its limit points, hence is closed. Conversely, suppose A is closed and S is complete. Let $\{x_n\}$ be a Cauchy sequence in S. Then $\{x_n\}$ converges to some limit x_0 in S. This x_0 is a limit point of A (since it is the limit of a sequence of points in A) so since A is closed, $x_0 \in A$. Ergo $\{x_n\}$ converges in A, implying that A is complete.

- Sets in $\mathbb{R}^2$. Drawing the sets shows that S_1 and S_2 are not bounded, and S_5 is not closed, so these three are not compact. However, S_3 and S_4 are closed and bounded in $\mathbb{R}^2$, hence compact.

- Metrics with the same convergence properties.

  - We have $B_{(\mathbb{R}^n, d_2)}(\mathbf{x}; \frac{r}{\sqrt{n}}) \subset B_{(\mathbb{R}^n, ||\cdot||)}(\mathbf{x}; r)$ (the box of edge length $\frac{2r}{\sqrt{n}}$ fits inside the sphere of radius r) and $B_{(\mathbb{R}^n, d_2)}(\mathbf{x}; r) \subset B_{(\mathbb{R}^n, ||\cdot||)}(\mathbf{x}; r)$ (the sphere of radius r fits inside the box of edge length $2r$).

  - Suppose $\mathbf{x}^k \to \mathbf{x}^0$ with respect to the metric d_2. Then for every $\epsilon > 0$, there is some N such that $k \ge N$ implies that $d_2(\mathbf{x}^k, \mathbf{x}^0) < \epsilon\sqrt{n}$. But by the first part of this problem, this implies that $||\mathbf{x}^n - \mathbf{x}^0|| < \epsilon$ for $n \ge N$, so $\mathbf{x}^k \to \mathbf{x}^0$ with respect to the metric d_2. Likewise, suppose $\mathbf{x}^k \to \mathbf{x}^0$ with respect to the metric $||\cdot||$. Then for every $\epsilon > 0$, there is some N such that $k \ge N$ implies that $||\mathbf{x}^k - \mathbf{x}^0|| < \epsilon$. But by the first part of this problem, this implies that $d_2(\mathbf{x}^k, \mathbf{x}^0) < \epsilon$ for $n \ge N$, so $\mathbf{x}^k \to \mathbf{x}^0$ with respect to the metric d_1.

- Extremely similar to the second part of this problem. Since Cauchy sequences converge in $\mathbb{R}^n, || \cdot ||)$, and since whether a sequence converges and whether a sequence is Cauchy is unaffected by the choice of metric, we conclude that $(\mathbb{R}^n, d_2)$ is complete.